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# A REMARK ON THE SCHRÖDINGER SMOOTHING EFFECT

by

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**Abstract.** — We prove the equivalence between the smoothing effect for a Schrödinger operator and the decay of the associate spectral projectors. We give two applications to the Schrödinger operator in dimension one.

**Résumé.** — On donne une caractérisation de l'effet régularisant pour un opérateur de Schrödinger par la décroissance de ses projecteurs spectraux. On en déduit deux applications à l'opérateur de Schrödinger en dimension un.

## 1. Introduction

Let  $d \geq 1$ , and consider the linear Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u = Hu, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = f(x) \in L^2(\mathbb{R}^d), \end{cases}$$

where  $H$  is a self-adjoint operator on  $L^2(\mathbb{R}^d)$ .

By the Hille-Yoshida theorem, the equation (1.1) admits a unique solution  $u(t) = e^{-itH}f \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ . Under suitable conditions on  $H$ , this solution enjoys a local gain of regularity (in the space variable) : For all  $T > 0$  there exists  $C > 0$  so that

$$\left( \int_0^T \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

for some weight  $\Psi$  and exponent  $\gamma > 0$ .

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This phenomenon has been discovered by T. Kato [7] in the context of KdV equations. For the Schrödinger equation in the case  $H = -\Delta$ , it has been proved by P. Constantin- J.-C. Saut [2], P. Sjölin [11], L. Vega [12] and K. Yajima [13]. The variable coefficients case has been obtained by S. Doi [3, 4, 5, 6].

The more general results are due to L. Robbiano-C. Zuily [9, 10] for equations with obstacles and potentials.

Let  $H$  be a self adjoint operator on  $L^2(\mathbb{R}^d)$ . It can be represented thanks to the spectral measure by

$$H = \int \lambda dE_\lambda.$$

In the sequel we moreover assume that  $H \geq 0$ . For  $N \geq 0$ , we can then define the spectral projector  $P_N$  associated to  $H$  by

$$(1.2) \quad P_N = \mathbf{1}_{[N, N+1[}(H) = \int \mathbf{1}_{[N, N+1[}(\lambda) dE_\lambda.$$

Our main result is a characterisation of the smoothing effect by the decay of the spectral projectors. Denote by  $\langle H \rangle = (1 + H^2)^{\frac{1}{2}}$ .

**Theorem 1.1 (Smoothing effect vs. decay).** —

Let  $\gamma > 0$  and  $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ . Then the following conditions are equivalent

(i) There exists  $C_1 > 0$  so that for all  $f \in L^2(\mathbb{R}^d)$

$$(1.3) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) There exists  $C_2 > 0$  so that for all  $N \geq 1$  and  $f \in L^2(\mathbb{R}^d)$

$$(1.4) \quad \|\Psi P_N f\|_{L^2(\mathbb{R}^d)} \leq C_2 N^{-\frac{\gamma}{2}} \|P_N f\|_{L^2(\mathbb{R}^d)}.$$

The interesting point is that we can take the same function  $\Psi$  and exponent  $\gamma > 0$  in both statements (1.3) and (1.4).

By the works cited in the introduction, in the case  $H = -\Delta$  on  $\mathbb{R}^d$ , (1.3) is known to hold with  $\gamma = \frac{1}{2}$  and  $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$ , for any  $\nu > 0$ .

There is also a class of operators  $H$  on  $L^2(\mathbb{R}^d)$  for which (1.3) is well understood. Let  $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ , and assume that for  $|x|$  large enough  $V(x) \geq C \langle x \rangle^k$  and that for any  $j \in \mathbb{N}^d$ , there exists  $C_j > 0$  so that  $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$ . Then L. Robbiano and C. Zuily [9] show that the smoothing effect (1.3) holds for the operator  $H = -\Delta + V(x)$ , with  $\gamma = \frac{1}{k}$  and  $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$ , for any  $\nu > 0$ .

We now turn to the case of dimension  $d = 1$ , and consider the operator  $H = -\Delta + V(x)$ . We make the following assumption on  $V$

**Assumption 1.** — *We suppose that  $V \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ , and that there exist  $2 < m \leq k$  so that for  $|x|$  large enough*

(i) *There exists  $C > 1$  so that  $\frac{1}{C}\langle x \rangle^k \leq V(x) \leq C\langle x \rangle^k$ .*

(ii)  *$V''(x) > 0$  and  $xV'(x) \geq mV(x) > 0$*

(iii) *For any  $j \in \mathbb{N}$ , there exists  $C_j > 0$  so that  $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$ .*

For instance  $V(x) = \langle x \rangle^k$  with  $k > 2$  satisfies Assumption 1.

It is well known that under Assumption 1, the operator  $H$  has a self-adjoint extension on  $L^2(\mathbb{R})$  (still denoted by  $H$ ) and has eigenfunctions  $(e_n)_{n \geq 1}$  which form an Hilbertian basis of  $L^2(\mathbb{R})$  and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geq 1,$$

with  $\lambda_n \rightarrow +\infty$ , when  $n \rightarrow +\infty$ .

For  $N \in \mathbb{N}$  the spectral projector  $P_N$  defined in (1.2) can be written in the following way. Let  $f = \sum_{n \geq 1} \alpha_n e_n \in L^2(\mathbb{R})$ , then

$$P_N f = \sum_{N \leq \lambda_n^2 < N+1} \alpha_n e_n.$$

Observe that we then have  $f = \sum_{N \geq 0} P_N f$ .

For such a potential, we can remove the spectral projectors in (1.4) and deduce from Theorem 1.1

**Corollary 1.2.** —

*Let  $\gamma > 0$  and  $\Psi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ . Let  $H = \Delta + V(x)$  so that  $V(x) = x^2$  or  $V(x)$  satisfies Assumption 1. Then the following conditions are equivalent*

(i) *There exists  $C_1 > 0$  so that for all  $f \in L^2(\mathbb{R})$*

$$(1.5) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R})}.$$

(ii) *There exists  $C_2 > 0$  so that for all  $n \geq 1$*

$$(1.6) \quad \|\Psi e_n\|_{L^2(\mathbb{R})} \leq C_2 \lambda_n^{-\gamma}, \quad \forall n \geq 1.$$

The statements (1.5) and (1.6) were obtained by K. Yajima & G. Zhang in [16] when  $\Psi$  is the indicator of a compact  $K \subset \mathbb{R}$  and with  $\gamma = \frac{1}{k}$ .

The statement (1.5) holds for  $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$ , by [9], but as far as we know, the bound (1.6) with this  $\Psi$  was unknown.

With Theorem 1.1 we are also able to prove the following smoothing effect for the usual Laplacian  $\Delta$  on  $\mathbb{R}$ .

**Proposition 1.3.** — *Let  $\Psi \in L^2(\mathbb{R})$ . Then there exists  $C > 0$  so that for all  $f \in L^2(\mathbb{R})$*

$$\left( \int_0^{2\pi} \|\Psi(x) \langle \Delta \rangle^{\frac{1}{4}} e^{-it\Delta} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C \|\Psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

From the works cited in the introduction, we have

$$\left( \int_{\mathbb{R}} \|\Psi(x) \langle \Delta \rangle^{\frac{1}{4}} e^{-it\Delta} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R})},$$

for  $\Psi(x) = \langle x \rangle^{\frac{1}{2}-\nu}$ , for any  $\nu > 0$ . Hence Proposition 1.3 shows that we can extend the class of the weights, but we are only able to prove local integrability in time.

**Notation.** — *We use the notation  $a \lesssim b$  if there exists a universal constant  $C > 0$  so that  $a \leq Cb$ .*

## 2. Proof of the results

We define the self adjoint operator  $A = [H]$  (entire part of  $H$ ) by

$$A = \int [\lambda] dE_\lambda.$$

Notice that we immediately have that  $A - H$  is bounded on  $L^2(\mathbb{R}^d)$ .

The first step in the proof of Theorem 1.1 is to show that we can replace  $e^{-itH}$  by  $e^{-itA}$  in (1.3)

**Lemma 2.1.** — *Let  $\gamma > 0$  and  $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ . Then the following conditions are equivalent*

(i) *There exists  $C_1 > 0$  so that for all  $f \in L^2(\mathbb{R}^d)$*

$$(2.1) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) *There exists  $C_2 > 0$  so that for all  $f \in L^2(\mathbb{R}^d)$*

$$(2.2) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_2 \|f\|_{L^2(\mathbb{R}^d)}.$$

*Proof.* — We assume (2.1) and we prove (2.2). Let  $f \in L^2(\mathbb{R}^d)$  and Define  $v = e^{-itH}f$ . This function solves the problem

$$(i\partial_t - A)v = (H - A)v, \quad v(0, x) = f(x).$$

Then by the Duhamel formula

$$\begin{aligned} e^{-itH}f = v &= e^{-itA}f - i \int_0^t e^{-i(t-s)A}(H - A)v \, ds \\ &= e^{-itA}f - i \int_0^{2\pi} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A}(H - A)v \, ds. \end{aligned}$$

Therefore by (2.1) and Minkowski

$$\begin{aligned} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH}v\|_{L_{2\pi}^2 L^2} &\lesssim \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA}v\|_{L_{2\pi}^2 L^2} \\ &\quad + \int_0^{2\pi} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A}(H - A)v\|_{L_t^2 L_x^2} \, ds \\ (2.3) \quad &\lesssim \|f\|_{L^2} + \int_0^{2\pi} \|(H - A)v\|_{L^2} \, ds. \end{aligned}$$

Now use that the operator  $(H - A) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is bounded, and by (2.3) we obtain

$$\|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH}v\|_{L_{2\pi}^2 L^2} \lesssim \|f\|_{L^2},$$

which is (2.2).

The proof of the converse implication is similar.  $\square$

*Proof of Theorem 1.1.* — The proof is based on Fourier analysis in time. This idea comes from [8] and has also been used in [16], but this proof was inspired by [1].

(i)  $\implies$  (ii) : To prove this implication, we use the characterisation (2.1). From (1.2) and the definition of  $A$ ,  $e^{-itA}P_N f = e^{-itN}P_N f$ . Hence it suffices to replace  $f$  with  $P_N f$  in (1.3) and (1.4) follows.

(ii)  $\implies$  (i) : Again we will use Lemma 2.1. We assume (2.2) and we first prove that

$$(2.4) \quad \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA}f\|_{L^2(0, 2\pi; L^2(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Write  $f = \sum_{N \geq 0} P_N f$ , then

$$\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA}f = \sum_{N \geq 0} e^{-iNt} \langle N \rangle^{\frac{\gamma}{2}} \Psi P_N f.$$

Now by Parseval in time

$$\|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi)}^2 \lesssim \sum_{N \geq 0} \langle N \rangle^\gamma |\Psi P_N f|^2,$$

and by integration in the space variable and (1.4)

$$\begin{aligned} \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))}^2 &\lesssim \sum_{N \geq 0} \langle N \rangle^\gamma \|\Psi P_N f\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim \sum_{N \geq 0} \|P_N f\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

which yields (2.4).

Now since the operator  $\langle A \rangle^{-\gamma/2} \langle H \rangle^{\gamma/2}$  is bounded on  $L^2$  and commutes with  $e^{-itA}$ , we have by (2.4)

$$\begin{aligned} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))} &= \\ &= \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} (\langle A \rangle^{-\frac{\gamma}{2}} \langle H \rangle^{\frac{\gamma}{2}} f)\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))} \\ &\lesssim \|\langle A \rangle^{-\frac{\gamma}{2}} \langle H \rangle^{\frac{\gamma}{2}} f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

which is (2.1). □

*Proof of Corollary 1.2.* — Let  $V$  satisfy Assumption 1. Then by [14, Lemma 3.3] there exists  $C > 0$  such that

$$|\lambda_{n+1}^2 - \lambda_n^2| \geq C \lambda_n^{1-\frac{2}{m}},$$

for  $n$  large enough. This implies that  $[\lambda_n^2] < [\lambda_{n+1}^2]$  for  $n$  large enough, because  $m > 2$  and  $\lambda_n \rightarrow +\infty$ . As a consequence

$$P_N f = \alpha_n e_n, \text{ with } n \text{ so that } N \leq \lambda_n^2 < N+1,$$

and this yields the result.

We now consider  $V(x) = x^2$ . In this case, the eigenvalues are the integers  $\lambda_n^2 = 2n+1$ , and the claim follows. □

**Remark 2.2.** — With this time Fourier analysis, we can prove the following smoothing estimate for  $H$  which satisfies Assumption 1

$$\|\langle H \rangle^{\frac{\theta(q,k)}{2}} e^{-itH} f\|_{L^p(\mathbb{R}; L^2(0,T))} \lesssim \|f\|_{L^2(\mathbb{R})},$$

where  $\theta$  is defined by

$$\theta(q, k) = \begin{cases} \frac{2}{k}(\frac{1}{2} - \frac{1}{q}) & \text{if } 2 \leq q < 4, \\ \frac{1}{2k} - \eta \text{ for any } \eta > 0 & \text{if } q = 4, \\ \frac{1}{2} - \frac{2}{3}(1 - \frac{1}{q})(1 - \frac{1}{k}) & \text{if } 4 < q < \infty, \\ \frac{4-k}{6k} & \text{if } q = \infty. \end{cases}$$

This was done in [16] with a slightly different formulation.

*Proof of Proposition 1.3.* — By Theorem 1.1, we have to prove that the operator  $T$  defined by

$$Tf(x) = N^{\frac{1}{4}}\Psi(x)\mathbf{1}_{[N, N+1]}(-\Delta)f(x),$$

is continuous from  $L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$  with norm independent of  $N \geq 1$ . By the usual  $TT^*$  argument, it is enough to show the result for  $TT^*$ .

The kernel of  $T$  is  $K(x, y) = N^{\frac{1}{4}}\Psi(x)F_N(x - y)$  where

$$(2.5) \quad F_N(u) = \frac{1}{2\pi} \int e^{iu\xi} \mathbf{1}_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi = 4 \cos(D_N u) \frac{\sin(C_N u)}{u},$$

with  $C_N = (\sqrt{N+1} - \sqrt{N})/2$  and  $D_N = (\sqrt{N+1} + \sqrt{N})/2$ .

The kernel of  $TT^*$  is given by

$$\Lambda(x, z) = \int K(x, y) \overline{K}(z, y) dy,$$

and by Parseval and (2.5)

$$\begin{aligned} \Lambda(x, z) &= N^{\frac{1}{2}}\Psi(x)\Psi(z) \int F_N(x - y) \overline{F_N}(z - y) dy \\ &= \frac{1}{4} N^{\frac{1}{2}}\Psi(x)\Psi(z) \int e^{i(x-z)\xi} \mathbf{1}_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi \\ &= \pi N^{\frac{1}{2}}\Psi(x)\Psi(z) \cos(D_N(x - z)) \frac{\sin(C_N(x - z))}{x - z}. \end{aligned}$$

Now, since  $C_N \lesssim 1/\sqrt{N}$  and  $|\sin(x)| \leq |x|$ , we deduce that  $|\Lambda(x, z)| \leq C|\Psi(x)||\Psi(z)|$  (independent of  $N \geq 1$ ), and  $TT^*$  is continuous for  $\Psi \in L^2(\mathbb{R})$ .  $\square$

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## References

- [1] N. Burq, G. Lebeau and F. Planchon. Global existence for energy critical waves in 3-D domains. *Journal of the A.M.S.*, (21) 831–845, 2008.
- [2] P. Constantin and J.-C. Saut. Local smoothing properties of dispersive equations. *Journal American Mathematical Society* (1988) 413–439.
- [3] S. Doi. Smoothing effects of Schrödinger evolution group on Riemannian manifolds. *Duke Math. J.* 82 (1996) 679–706.
- [4] S. Doi. Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow. *Math. Ann.* 318 (2000) 355–389.
- [5] S. Doi. Remarks on the Cauchy problem for Schrödinger type equations. *Comm. in PDE*, 21 (1996) 163–178.
- [6] S. Doi. Smoothness of solutions for Schrödinger equations with unbounded potential. *Publ. Res. Inst. Math. Sci* 41 (2005), 1, 175–221.
- [7] T. Kato. On nonlinear Schrödinger equations, II.  $H^s$ -solutions and unconditional well-posedness, *J. Anal. Math.*, 87 (1995) 281–306.
- [8] G. Mockenhaupt, A. Seeger and C. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.
- [9] L. Robbiano, and C. Zuily. Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials. *Comm. Partial Differential Equations*, 33 (2008), no. 4-6, 718–727.
- [10] L. Robbiano, and C. Zuily. The Kato Smoothing Effect for Schrödinger equations with unbounded potentials in exterior domains. *Int. Math. Res. Notices*, 15 (2009), 1636–1698.
- [11] P. Sjölin. Regularity of solution to the Schrödinger equation. *Duke Math. J.* 55 (1987) 699–715.
- [12] L. Vega. Schrödinger equations, pointwise convergence to the initial data. *Proc. Amer. Math. Soc.* 102 (1988) 874–878.
- [13] K. Yajima. On smoothing property of Schrödinger propagator. *Lectures notes in Math.* 1450 Springer Verlag (1990) 20–35.
- [14] K. Yajima. Smoothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations. *Comm. Math. Phys.* 180 (1996) no. 3, 605–629.
- [15] K. Yajima, and G. Zhang. Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. *J. Differential Equations* (2004), no. 1, 81–110.
- [16] K. Yajima, and G. Zhang. Smoothing property for Schrödinger equations with potential superquadratic at infinity. *Comm. Math. Phys.* 221 (2001), no. 3, 573–590.

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